ON THE THEORY OF INVARIANTS OF N-LINES

BY

LENNIE PHOEBE COPELAND

Thesis Presented to the Faculty of the Graduate School of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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ON THE THEORY OF INVARIANTS OF n-LINES.

BY LENNIE PHOEBE COPELAND.

The invariants of plane polygons or n-lines have been studied from various points of view by Morley,* Leib,† Hun,‡ Glenn,§ Loud|| and MacNeish.¶

The purpose of this paper is to construct the elements of the formal theory of the plane m-line, where the latter is assumed to be represented always by a ternary form whose coefficients are given in terms of the absolute minimum number of independent parameters. For a ternary form f_{3m} to represent an m-line this minimum number is $2m$ and all the coefficients of f_{3m} are then rational in the parameters.

In fact it has been shown by Glenn that if the $2m$ parameters be taken as the coefficients of $a_{0x}{}^m$, $a_{1x}{}^{m-1}$, where

$$f_{3m} = a_{0x}{}^m + x_3 a_{1x}{}^{m-1} + \cdots + x_3{}^m a_{m0}$$

and

$$a_{ix}{}^{m-i} \equiv a_{i0}x_1{}^{m-i} + a_{i1}x_1{}^{m-i-1}x_2 + a_{i2}x_1{}^{m-i-2}x_2{}^2 + \cdots + a_{i\,m-i}x_2{}^{m-i} \qquad (i = 0, 1, 2, \cdots m),$$

then the remaining coefficients of f_{3m} are expressible as simple rational invariants of $a_{0x}{}^m$, $a_{1x}{}^{m-1}$. In section (1) we consider seminvariantive sets of conditions in order that two 2-lines, or two 3-lines may be projectively related to a triangle, by certain apolar laws. In section (2) we study the theory of full invariants and complete systems of m-lines. Applications of this theory are given in section (3).

The present writer wishes to thank Professor O. E. Glenn, who suggested the undertaking of this investigation, for numerous helpful suggestions and criticisms.

* Morley, "On the metric properties of the plane n-line," Am. Math. Soc. Trans., vol. 1; "On the orthocentric properties of the plane n-line," Am. Math. Soc. Trans., vol. 4, 5; "On two cubic curves in triangular relations," London Math. Soc. Proceedings (2), 4.

† Leib, "On the invariants of two triangles," Am. Math. Soc. Trans., vol. 10.

‡ Hun, "Invariant relations of two triangles," Am. Math. Soc. Trans., vol. 5.

§ Glenn, "On semi-discriminants of ternary forms," Trans. of the Am. Math. Soc., vol. 12, pp. 367–374; "On the structure of forms and the algebraical theory of n-lines," Am. Journal of Mathematics, vol. 34, No. 4.

|| Loud, "Sundry metric theorems concerning n-lines in a plane," Am. Math. Soc. Trans., vol. 1.

¶ MacNeish, "Linear polars of k-hedrons in n-space," University of Chicago Press, 1912.

1. Seminvariants of f_{32}, f_{33}. The form of the ternary m-line quantic due to Glenn is

$$f_{3m} = x_2^m l_{0\,x_1/x_2} + x_2^{m-1} x_3 l_{1\,x_1/x_2} + D^{-1}(-1)^{\frac{1}{2}m(m-1)}$$

$$(1) \qquad \times \sum_{j=2}^{m} x_3^j \sum_{i=0}^{m-j} \frac{\Delta_1^{m-j-i} \Delta_2^i R_m}{(m-j-i)!\,i!} x_1^{m-j-i} x_2^i$$

$$\equiv \prod_{i=1}^{m} \left(x_1 + r_i x_2 - \frac{l_{1-r_i}}{l'_{0-r_i}} \right)$$

where

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}},$$

$$x_2^{m-k} l_{k\,x_1/x_2} = a_{k0} x_1^{m-k} + a_{k1} x_1^{m-k-1} x_2 + \cdots + a_{km-k} x_2^{m-k},$$

and R_m is the resultant of $x_2^m l_{0\,x_1/x_2}$ and $x_2^{m-1} l_{1\,x_1/x_2}$, and D the discriminant of $x_2^m l_{0\,x_1/x_2}$. Consider a pair of 2-lines

$$f_{32} = x_2^2 l_{0\,x_1/x_2} + x_2 x_3 l_{1\,x_1/x_2} - \frac{R_2}{D} x_3^2 = \alpha_{1x} \alpha_{2x} = 0,$$

$$g_{32} = x_2^2 \varphi_{0\,x_1/x_2} + x_2 x_3 \varphi_{1\,x_1/x_2} - \frac{R_2'}{D'} x_3^2 = \beta_{1x} \beta_{2x} = 0.$$

These will intersect the sides of a given triangle, chosen as the triangle of reference, in harmonic ranges provided only

$$C_1 = 2a_{00} b_{02} + 2a_{02} b_{00} - a_{01} b_{01} = 0,$$

$$(2) \qquad C_2 = 2a_{00} D R_2' + 2b_{00} D' R_2 + a_{10} b_{10} D D' = 0,$$

$$C_3 = 2a_{02} D R_2' + 2b_{02} D' R_2 + a_{11} b_{11} D D' = 0.$$

We can, by referring to (1) ($m = 3$), also obtain a set of necessary and sufficient conditions in order that two given triangles f_{33}, g_{33} may cut the respective sides of a third triangle (chosen as the triangle of reference) in apolar point ranges. These are

$$C_1' = 3a_{00} b_{03} - a_{01} b_{02} + a_{02} b_{01} - 3a_{03} b_{00} = 0,$$

$$(3) \qquad C_2' = 3a_{00} D R_3' - a_{10} D \Delta_1 R_3' + b_{10} D' \Delta_1 R_3 - 3b_{00} D' R_3 = 0,$$

$$C_3' = 3a_{03} D R_3' - a_{12} D \Delta_2 R_3' + b_{12} D' \Delta_2 R_3 - 3b_{03} D' R_3 = 0.$$

Similar sets of conditions are obtainable for the general case of two m-lines, directly from (1).

2. General Theory. Let the form f_{3m} (see (1)) be written as follows:

$$f_{3m} = \prod_{i=1}^{m} (r_{i1}x_1 + r_{i2}x_2 + r_{i3}x_3),$$

and let

$$a_{k_1k_2k_3} = \Sigma r_{11}r_{21} \cdots r_{k_11}r_{k_1+12}r_{k_1+22} \cdots r_{k_1+k_22}r_{k_1+k_2+13} \cdots r_{k_1+k_2+k_33},$$

where Σ is the ordinary elementary symmetric function of the three groups of homogeneous variables r. Then

$$(4)\quad \begin{aligned} \delta_{12}a_{k_1k_2k_3} &= \left(r_{11}\frac{\partial}{\partial r_{12}} + r_{21}\frac{\partial}{\partial r_{22}} + \cdots + r_{m1}\frac{\partial}{\partial r_{m2}}\right)a_{k_1k_2k_3} = (k_1+1)a_{k_1+1k_2-1k_3}, \\ \delta_{23}a_{k_1k_2k_3} &= (k_2+1)a_{k_1k_2+1k_3-1}, \text{ etc.} \end{aligned}$$

If any function of the roots r_{ij}, as $I(r)$, is expressible in terms of the coefficients

$$I(r) = J(a)$$

and if δ_{ij} annihilates I, then, as is readily shown by expanding $I(r)$ by Taylor's theorem,*

$$(5)\quad \Delta_{ij} = \sum_k \frac{\partial}{\partial a_{k_1k_2k_3}}(k_i+1)a_{\ldots k_i+1, \ldots, k_j-1, \ldots} \qquad (i, j = 1, 2, 3, i \neq j)$$

annihilates $J(a)$.

We proceed to show that any function $I(r)$ which has the annihilators δ_{ij} is a function of the determinants of the third order that can be formed from triads of the factors of f_{3m}.†

We assume

$$(6)\quad \delta_{12}I = \delta_{13}I = \delta_{21}I = \delta_{23}I = \delta_{31}I = \delta_{32}I = 0.$$

From these linear partial differential equations we obtain

$$\frac{dr_{12}}{r_{11}} = \frac{dr_{22}}{r_{21}} = \frac{dr_{32}}{r_{31}} = \cdots = \frac{dr_{m2}}{r_{m1}} = \frac{dI}{0},$$

$$\frac{dr_{13}}{r_{12}} = \frac{dr_{23}}{r_{22}} = \frac{dr_{33}}{r_{32}} = \cdots = \frac{dr_{m3}}{r_{m2}} = \frac{dI}{0},$$

$$\frac{dr_{11}}{r_{13}} = \frac{dr_{21}}{r_{23}} = \frac{dr_{31}}{r_{33}} = \cdots = \frac{dr_{m1}}{r_{m3}} = \frac{dI}{0}.$$

Employing the first three fractions in the first row of equalities and writing $\{r_{11}\,r_{22}\}$ for the complementary minor of the minor $|r_{11}r_{22}|$ in the determinant

$$d_{123} = |\,r_{11}r_{22}r_{33}|,$$

* Cf. Amer. Journal of Math., vol. 34, p. 450.

† Gordan: "Ueber Combinanten," Math. Annalen, vol. 5, p. 111.

we have

$$-r_{21}\{r_{11}r_{22}\}dr_{12}+r_{11}\{r_{11}r_{22}\}dr_{22}=0,$$

$$r_{31}\{r_{11}r_{32}\}dr_{12}-r_{11}\{r_{11}r_{32}\}dr_{32}=0,$$

$$-r_{31}\{r_{21}r_{32}\}dr_{22}+r_{21}\{r_{21}r_{32}\}dr_{32}=0.$$

Adding these and denoting by R_{ij} the co-factor of r_{ij} in d_{123},

(7) $$-R_{12}dr_{12}+R_{22}dr_{22}-R_{32}dr_{32}=0.$$

Likewise from the second and third rows respectively,

(8) $$R_{13}dr_{13}-R_{23}dr_{23}+R_{33}dr_{33}=0,$$

(9) $$R_{11}dr_{11}-R_{21}dr_{21}+R_{31}dr_{31}=0.$$

The sum of (7), (8), (9) gives

$$dd_{123}=0,$$

and hence we have the particular integral

$$d_{123}=C_{123}.$$

Similarly we may obtain C^{m_3} particular integrals

$$d_{ijk}=C_{ijk},$$

and I is a function of the determinants d_{ijk}, as stated.

Of course $I(r)$ is not necessarily rational in the "roots" r_{ij}, but we henceforth assume that I is homogeneous and rational in the determinants d_{ijk}, and either symmetric or alternating in the roots r_{ij}.

The above reasoning may be applied in case of a linearly factorable p-ary form, with similar results.

Following the analogy from the theory of binary invariants expressed in terms of the roots, let us now define a ternary root-difference as the expression obtained by expanding a determinant such as d_{123} on the first column and then dividing each term by the product of the elements of that column. Thus

$$A_{123}\equiv\frac{|r_{11}r_{22}r_{33}|}{r_{11}r_{21}r_{31}}=\frac{|r_{22}r_{33}|}{r_{21}r_{31}}-\frac{|r_{12}r_{33}|}{r_{11}r_{31}}+\frac{|r_{12}r_{23}|}{r_{11}r_{21}}$$

is a ternary root-difference.

We now prove the following theorem which introduces a type of invariant of a factorable ternary form which is also a function of differences:

Any homogeneous function I of the root-differences of a ternary m-line, which is such that in all products of differences of which it consists every root (row of determinant) is involved in the same number of factors, is an invariant of the form.

If every root is involved in the same number of factors in every term of I, the latter becomes upon multiplication by a power of the leading coefficient of the m-line's form a homogeneous function of the determinants d_{ijk}, and as each determinant is an invariant under the general ternary collineation, I will be an invariant. Next suppose that I does not fulfill the above condition, that is, every root does not occur in the same number of factors in the expression for the invariant I. Let us ass me for example $m = 4$ and

$$I = \Sigma A_{123} A_{124},$$

then

$$(r_{11} r_{21} r_{31} r_{41})^2 I = \Sigma r_{31} r_{41} (123)(124),$$

where

$$(123) = |r_{11} r_{22} r_{33}|, \quad \text{etc.}$$

to this apply the linear transformation of determinant -1

$$x_1 = -x_3', \qquad x_2 = x_2', \qquad x_3 = -x_1'.$$

then

$$f_{3m} = \prod_{i=1}^{4} (r_{i1} x_1 + r_{i2} x_2 + r_{i3} x_3),$$

$$f_{3m}' = \prod_{i=1}^{4} (-r_{i3} x_1' + r_{i2} x_2' - r_{i1} x_3'),$$

and we must have of course

$$\Sigma r_{31} r_{41} (123)(124) \equiv \pm \Sigma r_{33} r_{43} (123)(124),$$

which is obviously impossible. Evidently then only those functions are invariant which involve each root in the same number of factors.

Thus in general the form of an m-line invariant, which is also an invariant function of ternary root-differences is

$$(10) \qquad I = \Sigma C (r_1 r_2 r_3)^{a_1} (r_1 r_2 r_4)^{a_2} \cdots,$$

where each root r_i occurs in the same number of determinant factors in each term of Σ, and when expressed in terms of the coefficients I is annihilated by Δ_{ij}. A similar theory for contravariants is obtained by analogous reasoning.

It is to be noted that no requirement that I should be symmetric in the roots has been imposed. In fact it is often preferable to use an alternating function as an invariant rather than its symmetric square. For when the invariant is expressed in terms of the coefficients of the m-line's form it is often possible to remove an alternating factor, in which case the real invariant has its degree considerably diminished. Thus the invariant

condition that the triangle

$$f_{33} = \prod_{i=1}^{3} (x_1 + r_i x_2 - l_{1-r_i}/l'_{0-r_i}) = a_{0x}^3 + a_{1x}^2 x_3 + a_{2x} x_3^2 + a_{30} x_3^3$$

be a pencil is

$$Q = \begin{vmatrix} 1 & r_1 & l_{1-r_1}/l'_{0-r_1} \\ 1 & r_2 & l_{1-r_2}/l'_{0-r_2} \\ 1 & r_3 & l_{1-r_3}/l'_{0-r_3} \end{vmatrix}$$

$$= \sqrt{\Delta}[2a_{01}^2 a_{12} - a_{01}a_{02}a_{11} + 9a_{00}a_{03}a_{11} - 6a_{01}a_{03}a_{10} + 2a_{02}^2 a_{10} - 6a_{00}a_{02}a_{12}] = 0,$$

where Δ is the discriminant of the binary cubic a_{0x}^3. This is the most elementary full invariant of a plane m-line and is only of the third degree in the a's. Since it is of the first degree in the coefficients of a_{1x}^2, it follows that in the space of nine dimensions whose coördinates are all the coefficients of ternary forms of order three, the forms representing pencils fill a rational five dimensional spread included in the known rational spread* of six dimensions representing those forms which are linearly factorable.

When in a p-ary form $m = p$, the one full invariant of the m-line always has the form of a single determinant, and therefore the simplest full invariant of a binary quadratic is the condition for a double point; of a ternary 3-line, the condition for a triple point; of a quaternary quartic, the condition for a quadruple point, etc.

A 2-line has no full invariant, but it has a contravariant, representing its double point. Let

$$f_{32} = \prod_{i=1}^{2} (x_1 + r_i x_2 - l_{1-r_i}/l'_{0-r_i}) = 0$$

be the 2-line. Then its contravariant, readily computed by use of (1) and symmetric functions, is

$$U_2 = \sqrt{4a_{00}a_{02} - a_{01}^2}[(2a_{02}a_{10} - a_{01}a_{11})u_1 + (2a_{00}a_{11} - a_{01}a_{10})u_2 + (4a_{00}a_{02} - a_{01}^2)u_3] = 0.$$

We may readily derive the conditions that two 2-lines f_{32}, g_{32} (§ 1) form a harmonic pencil. In fact, if we abbreviate U_2, and the corresponding contravariant of g_{32} as

$$U_2 = \sqrt{D}[\lambda_1(a)u_1 + \lambda_2(a)u_2 + \lambda_3(a)u_3] = 0,$$
$$V_2 = \sqrt{D'}[\lambda(b)u_1 + \lambda_2(b)u_2 + \lambda_3(b)u_3] = 0,$$

and let C_1 be the seminvariant C_1 from set (2) § 1, the proposed conditions are the following three:

(11) $\lambda_1(a)\lambda_2(b) - \lambda_1(b)\lambda_2(a) = 0, \quad \lambda_1(a)\lambda_3(b) - \lambda_1(b)\lambda_3(a) = 0, \quad C_1 = 0.$

* Transactions Amer. Math. Soc., vol. 12, p. 368.

3. Special Complete Systems.—An invariant of

$$f_{3m} = \prod_{i=1}^{m} (r_{i1}x_1 + r_{i2}x_2 + r_{13}x_3)$$

has been shown to be a function of third order determinants of the general form

$$R = \Sigma(r_1r_2r_3)^{\alpha_1}(r_1r_2r_4)^{\alpha_2} \cdots (r_{m-2}r_{m-1}r_m)^{\alpha_m c_3},$$

where R is assumed to be integral and rational. We have proved concerning any homogeneous invariant function of ternary differences, I, that in every term of I each r_k occurs in exactly the same number of determinant factors as every other r. Under these defining conditions the number of distinct types of I will be finite. For they lead to a set of diophantine equations to be satisfied in positive integers:

$$\alpha_i + \alpha_j + \cdots = \alpha_{i1} + \alpha_{j1} + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and the totality of solutions of such a system is expressible in terms of a finite number.*

We proceed to special cases.

(i) $$m = 3.$$

Here the general form of R is

$$R_3 = (r_1r_2r_3)^{\alpha_1}$$

Hence every invariant of f_{33} is a power of

$$I_1 = (r_1r_2r_3)$$

which has been computed in terms of the coefficients. (See Q, § 2.)

The contravariants are

$$C_3 = \Sigma(r_1r_2u)^{\beta_1}(r_2r_3u)^{\beta_2}(r_3r_1u)^{\beta_3},$$

and from the defining conditions we have

$$\beta_1 + \beta_2 = \beta_1 + \beta_3 = \beta_2 + \beta_3.$$

Hence $\beta_1 = \beta_2 = \beta_3$ and the only contravariant is

$$I_2 = (r_1r_2u)(r_2r_3u)(r_3r_1u).$$

To interpret I_2 consider it as the product of three eliminants of pairs of factors of f_{33} taken with

$$u_x \equiv u_1x_1 + u_2x_2 + u_3x_3,$$

which is evidently the line equation of the vertices of f_{33}.

* The reader should consult Hilbert's paper on the binary case, Math. Annalen, vol. 33; also Cayley, Math. Annalen, vol. 34.

(ii) $$m = 4.$$

The general form of R for this case is

$$R_4 = \Sigma(r_1r_2r_3)^{\alpha_1}(r_1r_2r_4)^{\alpha_2}(r_1r_3r_4)^{\alpha_3}(r_2r_3r_4)^{\alpha_4}.$$

Hence $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Therefore every invariant of f_{34} is a power of

$$I_1 = (123)(124)(134)(234).$$

The contravariants are

$$C_4 = \Sigma(r_1r_2u)^{\beta_1}(r_1r_3u)^{\beta_2}(r_1r_4u)^{\beta_3}(r_2r_3u)^{\beta_4}(r_2r_4u)^{\beta_5}(r_3r_4u)^{\beta_6}.$$

Hence $\beta_4 = \beta_3$, $\beta_5 = \beta_2$, $\beta_6 = \beta_1$.
Therefore there are three contravariants

$$I_2 = \Sigma(r_1r_2u)(r_3r_4u),$$

$$I_3 = \Sigma(r_1r_2u)(r_1r_3u)(r_2r_4u)(r_3r_4u),$$

$$I_4 = (r_1r_2u)(r_1r_3u)(r_1r_4u)(r_2r_3u)(r_2r_4u)(r_3r_4u).$$

(iii) $$m = 5.$$

Here

$$R = \Sigma(123)^{\alpha_1}(124)^{\alpha_2}(125)^{\alpha_3}(134)^{\alpha_4}(135)^{\alpha_5}(145)^{\alpha_6}(234)^{\alpha_7}(235)^{\alpha_8}(245)^{\alpha_9}(345)^{\alpha_{10}}.$$

and we are able to express all the α's in terms of six independent ones. In fact

$$R = \Sigma(123)^{1/2(2\alpha_6-\alpha_7-\alpha_8+\alpha_9+\alpha_{10})}(124)^{1/2(2\alpha_5-\alpha_7+\alpha_8-\alpha_9+\alpha_{10})}(125)^{1/2(-2\alpha_5-2\alpha_6+3\alpha_7+\alpha_8+\alpha_9+\alpha_{10})}$$
$$(134)^{-\alpha_5-\alpha_6+\alpha_7+\alpha_8+\alpha_9}(135)^{\alpha_5}(145)^{\alpha_6}(234)^{\alpha_7}(235)^{\alpha_8}(245)^{\alpha_9}(345)^{\alpha_{10}}.$$

Taking $\alpha_i = 0, 1, 2$ we obtain the following invariants:

$$I_1 = \Sigma(123)(125)(134)(245)(345),$$
$$I_2 = \Sigma(125)(134)^2(235)(245),$$
$$I_3 = \Sigma(125)^3(134)^3(234)(235)(245)(345),$$
$$I_4 = \Sigma(123)(125)^2(134)^2(145)(234)(235)(245)(345),$$
$$I_5 = (123)(124)(125)(134)(135)(145)(234)(235)(245)(345),$$
$$I_6 = \Sigma(123)(124)(125)^3(134)^4(235)^2(245)^2(345)^2,$$
$$I_7 = \Sigma(123)^3(124)(125)(134)^2(145)^2(235)^2(245)^2(345)^2,$$
$$I_8 = \Sigma(123)(124)(125)(134)^2(135)^2(145)^2(234)^2(235)^2(245)^2,$$
$$I_9 = \Sigma(123)^3(125)(134)(145)(245)^2(345)^2,$$
$$I_{10} = \Sigma(123)^2(124)(125)(134)(135)(245)^2(345)^2.$$

Through the medium of (1) § 1 all of these invariants and covariants have expressions in terms of the actual coefficients.

PHILADELPHIA, PA.
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